Dispersive treatment of $K \to \pi\pi$

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Outline

- Soft pion theorems
- Dispersion relations
- Numerical solution
- Summary and discussion

Work done in collaboration with G. Colangelo, J. Kambor and F. Orellana, University of Zurich

Soft-pion theorem

$K \to \pi\pi$ amplitude:

$$_{I=0}\langle \pi(p_1)\pi(p_2)|\mathcal{H}_W^{1/2}(0)|K(q_1)\rangle =: T^+(s,t,u)$$

$$s = (p_1+p_2)^2, \quad t = (q_1-p_1)^2, \quad u = (q_1-p_2)^2 \quad ,$$
$$s+t+u = 2M_\pi^2 + M_K^2 + q_2^2$$

q_2 is the momentum carried by the weak Hamiltonian.

Physical amplitude:

$$\mathcal{A}(K \to \pi\pi) = T^+(M_K^2, M_\pi^2, M_\pi^2)$$

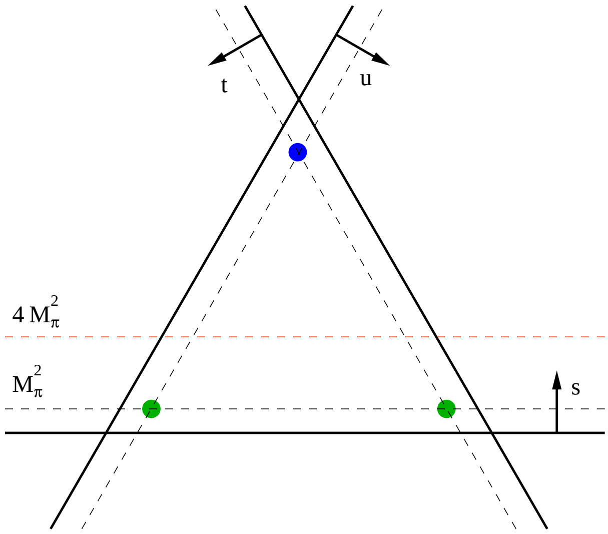
Soft-pion theorem:

$$T^+(M_\pi^2, M_K^2, q_2^2) = \frac{-1}{2F_\pi} \underbrace{\langle \pi(p_2)|\mathcal{H}_W^{1/2}(0)|K(q_1)\rangle}_{F_W(q_2^2)} + \mathcal{O}(M_\pi^2)$$

From now on $q_2^2 = 0$.

The theorem is based on:

- the chiral symmetry $SU(2)_L \times SU(2)_R$;
- the approximation $M_\pi = 0$ for $\pi(p_1)$
  $\Rightarrow$ expect corrections of order $M_\pi^2/(1\,\mathrm{GeV})^2 \sim 1\%$.



In practice, to get the physical amplitude one uses:

$$T^+(M_K^2, M_\pi^2, M_\pi^2) = \frac{-2}{F_\pi} F_W\,(0)$$

The relation is valid to leading order in chiral perturbation theory $(SU(3)_L \times SU(3)_R)$

$\Rightarrow$ expect corrections of order $M_K^2/(1\,\mathrm{GeV})^2 \sim 25\%$.

Decomposition of $T^+(s,t,u)$

Neglect the imaginary parts of D waves and higher

$$\Downarrow$$

$$\begin{aligned} T^+(s,t,u) &= M_0(s) + \left\{ \frac{1}{3}\left[N_0(t) + 2R_0(t)\right] \right. \\ &+ \left. \frac{1}{2}\left(s - u - \frac{M_\pi^2 \Delta}{t}\right) N_1(t) \right\} + \{t \leftrightarrow u\} \end{aligned}$$

where $\Delta = M_K^2 - M_\pi^2$.

The functions $M_0(s)$, $N_{0,1}(t)$ and $R_0(t)$ are defined to have only a right-hand cut:

$$\begin{aligned} \mathrm{disc} M_0(s) &= \sin\delta_0^0(s) e^{-i\delta_0^0}\left[M_0(s) + \hat{M}_0(s)\right] \\ \mathrm{disc} N_\ell(s) &= \sin\delta_\ell^{1/2}(s) e^{-i\delta_\ell^{1/2}}\left[N_\ell(s) + \hat{N}_\ell(s)\right] \\ \mathrm{disc} R_0(s) &= \sin\delta_0^{3/2}(s) e^{-i\delta_0^{3/2}}\left[R_0(s) + \hat{R}_0(s)\right] \quad , \end{aligned}$$

Angular averages

$$
\begin{aligned}
\hat{M}_0(s) &= \langle \tilde{N}_1 \rangle \left[-2M_K^2 M_\pi^2 + s\Sigma_1 - \frac{1}{4}(M_K^4 + 3s^2) \right] \\
&+ \langle z\tilde{N}_1 \rangle s|\mathbf{p}||\mathbf{q}| + \langle z^2 \tilde{N}_1 \rangle 4|\mathbf{p}|^2|\mathbf{q}|^2 \\
&+ \frac{2}{3}\langle N_0 \rangle + \frac{4}{3}\langle R_0 \rangle
\end{aligned}
$$

where

$$
\langle z^n X \rangle(s) = \frac{1}{2}\int_{-1}^{1} dz z^n X(\Sigma_1/2 - s/2 + 2|\mathbf{p}||\mathbf{q}|z)
$$

$$
\Sigma = M_K^2 + M_\pi^2 \;, \qquad \Sigma_1 = \Sigma + M_\pi^2 \qquad \tilde{N}_1(t) = \frac{N_1(t)}{t}
$$

Dispersion relations

$$M_0(s) = \Omega_0^0(s,s_0)\left\{a + b(s-s_0) + \frac{(s-s_0)^2}{\pi}\int_{4M_\pi^2}^{\Lambda_1^2} \frac{\sin\delta_0^0(s')\hat{M}_0(s')ds'}{|\Omega_0^0(s',s_0)|(s'-s)(s'-s_0)^2}\right\}$$

$$N_0(s) = \Omega_0^{1/2}(s)\left\{\frac{s^2}{\pi}\int_{(M_K+M_\pi)^2}^{\Lambda_2^2} \frac{\sin\delta_0^{1/2}(s')\hat{N}_0(s')ds'}{|\Omega_0^{1/2}(s')|(s'-s)s'^2}\right\}$$

where the Omnès functions are defined as:

$$\Omega_0^0(s,s_0) = \exp\left\{\frac{(s-s_0)}{\pi}\int_{4M_\pi^2}^{\tilde{\Lambda}_1^2} ds' \frac{\delta_0^0(s')}{(s'-s_0)(s'-s)}\right\}$$

$$\Omega_\ell^I(s) = \exp\left\{\frac{s}{\pi}\int_{(M_K+M_\pi)^2}^{\tilde{\Lambda}_2^2} ds' \frac{\delta_\ell^I(s')}{s'(s'-s)}\right\} \qquad I = \frac{1}{2}, \frac{3}{2}$$

Determination of the subtraction constants

Choosing $s_0 = M_\pi^2$ as subtraction point for $M_0(s)$ one can get a from the soft-pion theorem:

$$-\frac{1}{2F_\pi^2} F_W(0) = a + \frac{1}{3}\left[N_0(M_K^2) + 2R_0(M_K^2)\right] + O(M_\pi^2)$$

The other subtraction constant b can be related to the derivative of the amplitude at the soft-pion point:

$$b = \frac{\partial}{\partial s} T^+(s, \Sigma - s, M_\pi^2)_{|s=M_\pi^2} + \dots$$

There is a Ward identity for this derivative:

$$\frac{\partial}{\partial s} T^+(s, \Sigma - s, M_\pi^2)_{|s=M_\pi^2} = \frac{1}{2} C(M_\pi^2, M_K^2, M_\pi^2) + \mathcal{O}(M_\pi^2)$$

where

$$\frac{i}{F_\pi} \int dx e^{ip_1 x} \langle \pi(p_2) | T \mathcal{H}_W^{1/2}(0) A^\mu(x) | K(q_1) \rangle =$$

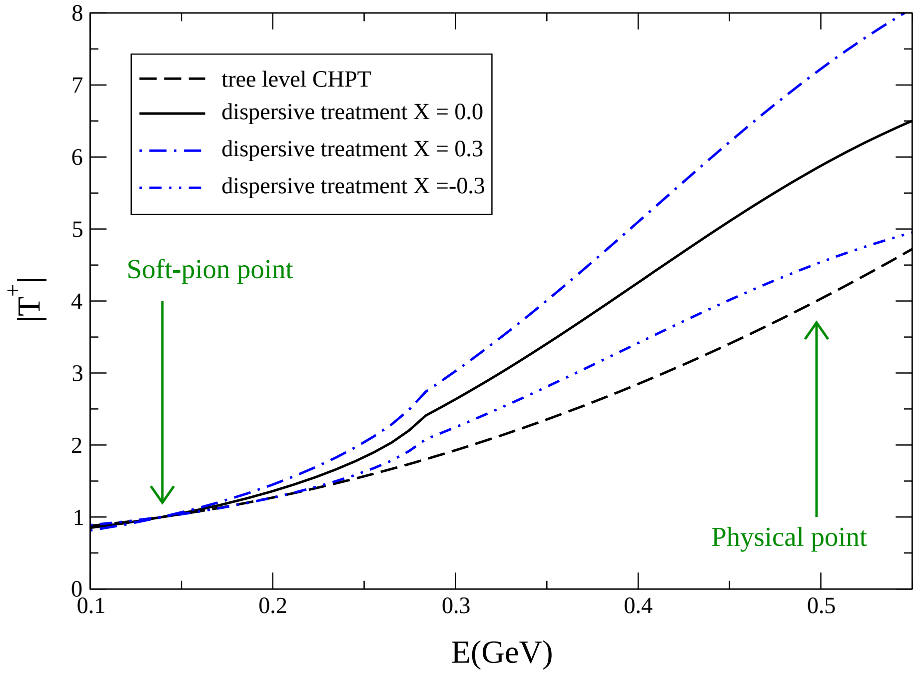
$$ip_1^\mu B + iq_1^\mu C + iq_2^\mu D$$

Numerical study of the dispersion relation

We use the following CHPT relation between a and b:

$$b = \frac{3a}{M_K^2 - M_\pi^2}\left(1 + X + O(M_K^4)\right)$$

and vary the correction to the leading order according to the rule of thumb $X \sim O(M_K^2) \sim \pm 30\%$

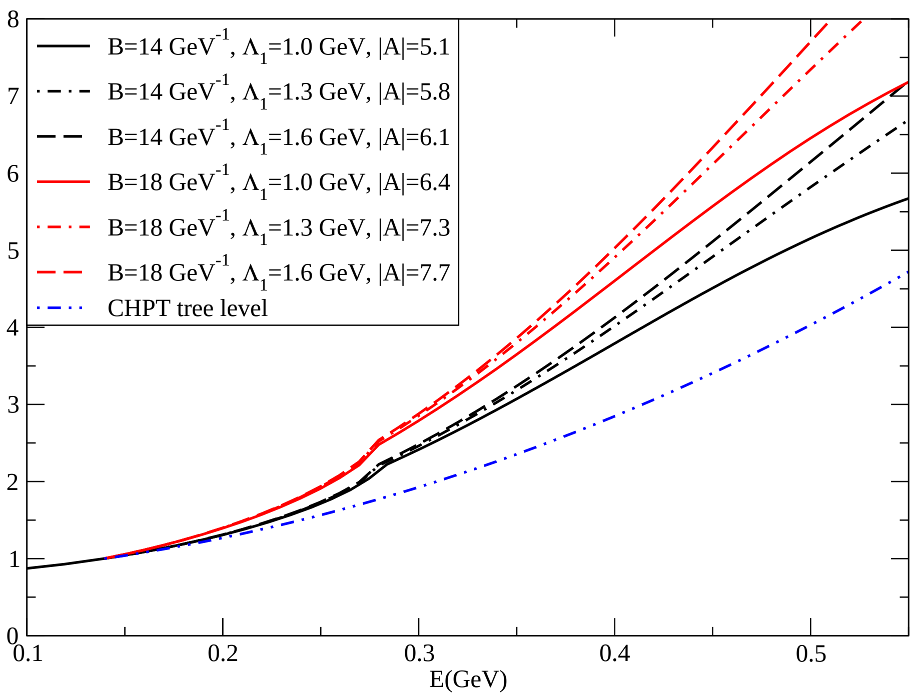


Dependence on the cutoff

Leading order in CHPT:

$$B = \frac{b}{a} = \frac{3}{M_K^2 - M_\pi^2} = 13\,\mathrm{GeV}^{-1}$$

The numerical solution of the dispersion relation depends mostly on B and Λ_1:



The dependence on the cutoff must be brought under control $\Rightarrow$ Coupled channel analysis!

Summary and discussion

- $\pi\pi$ final state interactions are important for the $K \to \pi\pi$ amplitude.
  (Truong (88), Kambor, Missimer and Wyler (91), Bertolini, Eeg and Fabbrichesi (96), Pallante, Pich and Scimemi (00-01));

- the presented method provides a clean framework to account for final state interactions;

- so far lattice QCD has provided the $K \to \pi$ using leading order CHPT to extrapolate to the $K \to \pi\pi$ matrix element. The above method is a clear improvement over this approximation.