Late time properties of a decaying false vacuum

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1. Introduction

A discussion of the decay of the false vacuum was began in two pioneer papers by Coleman and, Callan and Coleman

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[1] S. Coleman, Phys. Rev. D 15, 2929 (1977),
[2] C.G. Callan and S. Coleman, Phys. Rev. D 16, 1762 (1977).

"*The fate of the false vacuum*" was discussed there, namely the unstability of a physical system in a state which is not an absolute minimum of its energy density, and which is separated from the minimum by an effective potential barrier. It was shown, in those papers, that even if the state of the early Universe is too cold to activate a "*thermal*" transition (via thermal fluctuations) to the lowest energy (i.e. "*true vacuum*") state, a quantum decay from the false vacuum to the true vacuum may still be possible through a barrier penetration via macroscopic quantum tunneling.

Not long ago, the decay of the false vacuum state in a cosmological context has attracted renewed interest, especially in view of its possible relevance in the process of tunneling among the many vacuum states of the string landscape (a set of vacua in the low energy approximation of string theory), see e.g.:

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[3] L. M. Krauss, J. Dent, and G. D. Starkman, Int. J. Mod. Phys. D 17, 2501 (2008),

[4] Yung-Son Piao, arXiv:0810.3654

In many models the scalar field potential driving inflation has a multiple, low–energy minima or "*false vacuua*". Then the absolute minimum of the energy density is the "*true vacuum*".

In my talk the attention will not be focussed on inflationary processes and mechanisms to produce the various states in the landscape etc., **but on properties of decaying false vacuum states from the point of view of the quantum theory of unstable states evolving in time and decaying.**

Since the work of Khalfin

[5] L. A. Khalfin, Zh. Eksp. Teor. Fiz. **33**, 1371 (1957)[Sov. Phys. JETP **6**, 1053 (1958)]

it is known that for long times compared to the characteristic decay time of an unstable state (when the decay law has an exponential form), the survival probability of such states is no longer described by an exponential function of time t but it decreases as $t \to \infty$ more slowly than any exponential function of t.

Krauss and Dent analyzing a false vacuum decay

[6] L. M. Krauss, J. Dent, Phys. Rev. Lett., **100**, 171301 (2008); see also: S. Winitzki, Phys. Rev. **D 77**, 063508 (2008),

pointed out that in eternal inflation, even though regions of false vacua by assumption should decay exponentially, gravitational effects force space in a region that has not decayed yet to grow exponentially fast. This effect causes that many false vacuum regions can survive up to the times much later than times when the exponential decay law holds. In the mentioned paper by Krauss and Dent the attention was focused on the possible behavior of the unstable false vacuum at very late times, where deviations from the exponential decay law become to be dominant.

The aim of this talk is to discuss the late time behavior of the energy of the false vacuum states.

2. Properties of unstable states in short

If $|M\rangle$ is an initial unstable state then the survival probability, $\mathcal{P}(t)$, equals

$$\mathcal{P}(t) = |a(t)|^2,$$

where $a(t)$ is the survival amplitude,

$$a(t) = \langle M|M;t\rangle, \quad \text{and} \quad a(0) = 1,$$

and

$$|M;t\rangle = e^{-i\,\frac{t}{\hbar}H}\,|M\rangle,$$

H is the total Hamiltonian of the system under considerations.

The spectrum, $\sigma(H)$, of H is assumed to be bounded from below, $\sigma(H) = [E_{min}, \infty)$ and $E_{min} > -\infty$.

From basic principles of quantum theory it is known that the amplitude $a(t)$, and thus the decay law $\mathcal{P}_M(t)$ of the unstable state $|M\rangle$, are completely determined by the density of the energy distribution function $\omega(\mathcal{E})$ for the system in this state

$$a(t) = \int_{Spec.(H)} \omega(E)\, e^{-\frac{i}{\hbar}\, E\, t}\, dE. \tag{1}$$

where

$$\omega(E) \geq 0 \quad \text{and} \quad \omega(E) = 0 \quad \text{for} \quad E < E_{min}.$$

From this last condition and from the Paley–Wiener Theorem it follows that there must be (see [5])

$$|a(t)| \;\geq\; A\, e^{-b\, t^{q}},$$

for $|t| \rightarrow \infty$. Here $A > 0$, $b > 0$ and $0 < q < 1$.
This means that the decay law $\mathcal{P}_M(t)$ of unstable states decaying in the vacuum can not be described by an exponential function of time t if time t is suitably long, $t \rightarrow \infty$, and that for these lengths of time $\mathcal{P}_\phi(t)$ tends to zero as $t \rightarrow \infty$ more slowly than any exponential function of t.

The analysis of the models of the decay processes shows that

$$\mathcal{P}_M(t) \simeq e^{-\frac{\Gamma_M t}{\hbar}},$$

(where Γ_M is the decay rate of the state $|M\rangle$), to a very high accuracy at the canonical decay times t: From t suitably later than the initial instant t_0 up to

$$t \gg \tau_M = \frac{\hbar}{\Gamma_M}$$

and smaller than $t = T$, where T is the crossover time and denotes the time t for which the non–exponential deviations of $a(t)$ begin to dominate.

In general, in the case of quasi–stationary (metastable) states it is convenient to express $a(t)$ in the following form

$$a(t) = a_{exp}(t) + a_{non}(t), \qquad (2)$$

where $a_{exp}(t)$ is the exponential part of $a(t)$, that is

$$a_{exp}(t) = N\, e^{-i\, \frac{t}{\hbar}(E_M - \frac{i}{2}\, \Gamma_M)}, \tag{3}$$

(E_M is the energy of the system in the state $|M\rangle$ measured at the canonical decay times, N is the normalization constant), and $a_{non}(t)$ is the non–exponential part of $a(t)$.

For times $t \sim \tau_M$:

$$|a_{exp}(t)| \gg |a_{non}(t)|,$$

The crossover time T can be found by solving the following equation,

$$|a_{exp}(t)|^{\,2} = |a_{non}(t)|^{\,2}. \tag{4}$$

The amplitude $a_{non}(t)$ exhibits inverse power–law behavior at the late time region: $t \gg T$.

Indeed, the integral representation (1) of $a(t)$ means that $a(t)$ is the Fourier transform of the energy distribution function $\omega(E)$. Using this fact we can find asymptotic form of $a(t)$ for $t \to \infty$. Results are rigorous.

Let us assume now that $\lim_{E \to E_{min}+} \, \omega(E) \stackrel{\text{def}}{=} \omega_0 > 0$. Let derivatives $\omega^{(k)}(E)$, $(k = 0, 1, 2, \ldots, n)$, be continuous in $[E_{min}, \infty)$, (that is let for $E > E_{min}$ all $\omega^{(k)}(E)$ be continuous and all the limits $\lim_{E \to E_{min}+} \, \omega^{(k)}(E)$ exist) and let all these $\omega^{(k)}(E)$ be absolutely integrable functions then

$$a(t) \underset{t \to \infty}{\sim} -\frac{i\hbar}{t} \, e^{-\frac{i}{\hbar} E_{min} t} \sum_{k=0}^{n-1} (-1)^k \left(\frac{i\hbar}{t}\right)^k \omega_0^{(k)} = a_{non}(t), \tag{5}$$

where $\omega_0^{(k)} \stackrel{\text{def}}{=} \lim_{E \to E_{min}+} \, \omega^{(k)}(E)$.

[7] K. Urbanowski, *Eur. Phys. J. D*, **54**, (2009),

Let us now consider a more complicated form of the density $\omega(E)$. Namely let $\omega(E)$ be of the form

$$\omega(E) = (E - E_{min})^{\lambda}\, \eta(E) \;\in\; L_1(-\infty, \infty), \qquad (6)$$

where $0 < \lambda < 1$ and it is assumed that $\eta(E_{min}) > 0$ and $\eta^{(k)}(E)$, $(k = 0, 1, \ldots, n)$, exist and they are continuous in $[E_{min}, \infty)$, and limits $\lim_{E \to E_{min}+}\; \eta^{(k)}(E)$ exist, $\lim_{E \to \infty}\, (E - E_{min})^{\lambda}\, \eta^{(k)}(E) = 0$ for all above mentioned k, then

$$a(t) \underset{t\to\infty}{\sim} (-1)\, e^{-\frac{i}{\hbar} E_{min} t} \Big[\Big(-\frac{i\hbar}{t} \Big)^{\lambda+1} \Gamma(\lambda+1)\, \eta_0 + \lambda \Big(-\frac{i\hbar}{t} \Big)^{\lambda+2} \Gamma(\lambda+2)\, \eta_0^{(1)} + \ldots \Big] = a_{non}(t) \qquad (7)$$

3. Instantaneous energy and instantaneous decay rate

The amplitude $a(t)$ contains information about the decay law $\mathcal{P}_M(t)$ of the state $|M\rangle$, that is about the decay rate Γ_M of this state, as well as the energy E_M of the system in this state. This information can be extracted from $a(t)$. Indeed if $|M\rangle$ is an unstable (a quasi–stationary) state then

$$a(t) \cong e^{-\frac{i}{\hbar}(E_M - \frac{i}{2}\Gamma_M)\, t}. \tag{8}$$

So, there is

$$E_M - \frac{i}{2}\Gamma_M \equiv i\hbar \frac{\partial a(t)}{\partial t}\, \frac{1}{a(t)}, \tag{9}$$

in the case of quasi–stationary states.

The standard interpretation and understanding of the quantum theory and the related construction of our measuring devices are such that detecting the energy E_M and decay rate Γ_M one is sure that the amplitude $a(t)$ has the form (8) and thus that the relation (9) occurs.

Taking the above into account one can define the "effective Hamiltonian", h_M, for the one–dimensional subspace of states $\mathcal{H}_{||}$ spanned by the normalized vector $|M\rangle$ as follows

$$h_M \stackrel{\rm def}{=} i\hbar \frac{\partial a(t)}{\partial t} \; \frac{1}{a(t)}. \tag{10}$$

In general, h_M can depend on time t, $h_M \equiv h_M(t)$. One meets this effective Hamiltonian when one starts with the Schrödinger Equation for the total state space $\mathcal{H}$ and looks for the rigorous evolution equation for the distinguished subspace of states $\mathcal{H}_{||} \subset \mathcal{H}$.

The equivalent expression for $h_M \equiv h_M(t)$ has the following form [7]

$$h_M(t) \equiv \frac{\langle M|H|M;t\rangle}{\langle M|M;t\rangle} \stackrel{\rm def}{=} \mathcal{E}_M(t) - \frac{i}{2}\,\gamma_M(t). \tag{11}$$

Details can be found in [7] and in

[8] K. Urbanowski, Cent. Eur. J. Phys. **7**, (2009), (see also references one can find therein).

Thus, one finds the following expressions for the energy and the decay rate of the system in the state $|M\rangle$ under considerations, to be more precise for the instantaneous energy $\mathcal{E}_M(t)$ and the instantaneous decay rate, $\gamma_M(t)$,

$$\mathcal{E}_M \equiv \mathcal{E}_M(t) = \Re\,(h_M(t)), \tag{12}$$

$$\gamma_M \equiv \gamma_M(t) = -\,2\,\Im\,(h_M(t)), \tag{13}$$

where $\Re\,(z)$ and $\Im\,(z)$ denote the real and imaginary parts of z respectively.

Using (10) and (19), (20) one can find that

$$\mathcal{E}_M(0) = \langle M|H|M\rangle, \tag{14}$$

$$\mathcal{E}_M(t \sim \tau_M) \simeq E_M \neq \mathcal{E}_M(0), \tag{15}$$

$$\gamma_M(0) = 0, \tag{16}$$

$$\gamma_M(t \sim \tau_M) \simeq \Gamma_M. \tag{17}$$

So, there is $\mathcal{E}_M(t) = E_M$ at the canonical decay time.

Starting from the asymptotic expressions (5) and (7) for $a(t)$ and using (10) after some algebra one finds for times $t \gg T$ that

$$h_M(t)\big|_{t\to\infty} \simeq E_{min} + (-\frac{i\hbar}{t})\, c_1 + (-\frac{i\hbar}{t})^2\, c_2 + \ldots, \quad (18)$$

where $c_i = c_i^*, \quad i = 1, 2, \ldots$; (coefficients c_i depend on $\omega(E)$).
This last relation means that

$$\mathcal{E}_M(t) \simeq E_{min} + \frac{c_2}{t^2} \ldots, \quad (\text{for } t \gg T), \quad (19)$$

$$\gamma_M(t) \simeq 2\,\frac{c_1}{t} + \ldots, \quad (\text{for } t \gg T), \quad (20)$$

These properties take place for all unstable states which survived up to times $t \gg T$.
From (19) it follows that $\lim_{t\to\infty} \mathcal{E}_M(t) = E_{min}$.

For the most general form (6) of the density $\omega(E)$ (i. e. for $a(t)$ having the asymptotic form given by (7)) we have

$$c_1 = \lambda + 1, \quad c_2 = (\lambda + 1)\,\frac{\eta^{(1)}(E_{min})}{\eta(E_{min})}. \qquad (21)$$

The energy densities $\omega(E)$ considered in quantum mechanics and in quantum field theory can be described by $\omega(E)$ of the form (6), eg. quantum field theory models correspond with $\lambda = \frac{1}{2}$.
The average energy measured at some time interval (t_1, t_2) (with $t_1, t_2 \gg T$) equals

$$\overline{\mathcal{E}_M(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mathcal{E}_M(t)\, dt \simeq E_{min} + \frac{c_2}{t_1\, t_2} + \ldots, \qquad (22)$$

4. Discussion

Krauss and Dent in their paper [6] mentioned earlier made a hypothesis that some false vacuum regions do survive well up to the time T or later. Let E_0^{false} be the energy of a state corresponding to the false vacuum measured at the canonical decay time and E_0^{true} be the energy of true vacuum (i.e. the true ground state of the system). As it is seen from the results presented in previous Section, the problem is that the energy of those false vacuum regions which survived up to T and much later differs from E_0^{false},

[9] K. Urbanowski, *Phys. Rev. Lett.*, **107**, 209001 (2011), (see also references one can find therein).

So, if one assumes that $E_0^{true} \equiv E_{min}$ then one has for the false vacuum state that at $t \gg T$

$$E_0^{false}(t) \simeq E_0^{true} + \frac{c_2}{t^2} \ldots \neq E_0^{false}. \tag{23}$$

Similarly

$$\gamma_0^{false}(t) \simeq +2\,\frac{c_1}{t} \ldots \quad (\text{for } \; t \gg T). \tag{24}$$

Two last properties of the false vacuum states mean that

$$E_0^{false}(t) \to E_0^{true} \;\text{ and }\; \gamma_0^{false}(t) \to 0 \;\text{ as }\; t \to \infty, \tag{25}$$

5. Final remarks

The basic physical factor forcing the wave function $|M; t\rangle$ and thus the amplitude $a(t)$ to exhibit inverse power law behavior at $t \gg T$ is a boundedness from below of $\sigma(H)$. This means that if this condition takes place and

$$\int_{-\infty}^{+\infty} \omega(E)\, dE \,<\, \infty, \tag{26}$$

then all properties of $a(t)$, including a form of the time–dependence at $t \gg T$, are the mathematical consequence of them both. The same applies by (10) to properties of $h_M(t)$ and concerns the asymptotic form of $h_M(t)$ and thus of $\mathcal{E}_M(t)$ and $\gamma_M(t)$ at $t \gg T$.

Note that properties of $a(t)$ and $h_M(t)$ discussed above do not take place when $\sigma(H) = (-\infty, +\infty)$.

- The late time behavior of the energy of the system in the false vacuum state,

$$E_0^{false}(t) \simeq E_0^{true} + \frac{c_2}{t^2} \ldots, \quad \text{for} \quad t \gg T, \tag{27}$$

is the pure quantum effect following from the basic principles of the quantum theory.

- **Problem:** Do properties (23) – (25) of the false vacuum states (i.e. changes in time of $E_0^{false}(t)$ at $t \gg T$ – see (27)) affect time variations of fundamental constants?

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[10] J. K. Webb, *et al.*, Phys. Rev. Lett., **107**, 191101, 2011;
J. C. Berengut, V. V. Flambaum, arXiv:1008.3957;
J. P. Uzan, Rev. Mod. Phys. **75**, 403 (2003),
John N. Bahcall, *et al.*, The Astrophysical Journal, **600**,520, 2004.

Taking into account the above discussed late time properties of the false vacuum and possible variations in time of fundamental constants the following question can arise:

Are we living in the Universe with a false vacuum?

Thank you for your attention