Statistical issues for future neutrino experiments

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1 The usual deal

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2 Frequentist methods

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3 Bayesian methods

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4 Summary and conclusions

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4 Summary and conclusions

Parameter estimation sensitivity

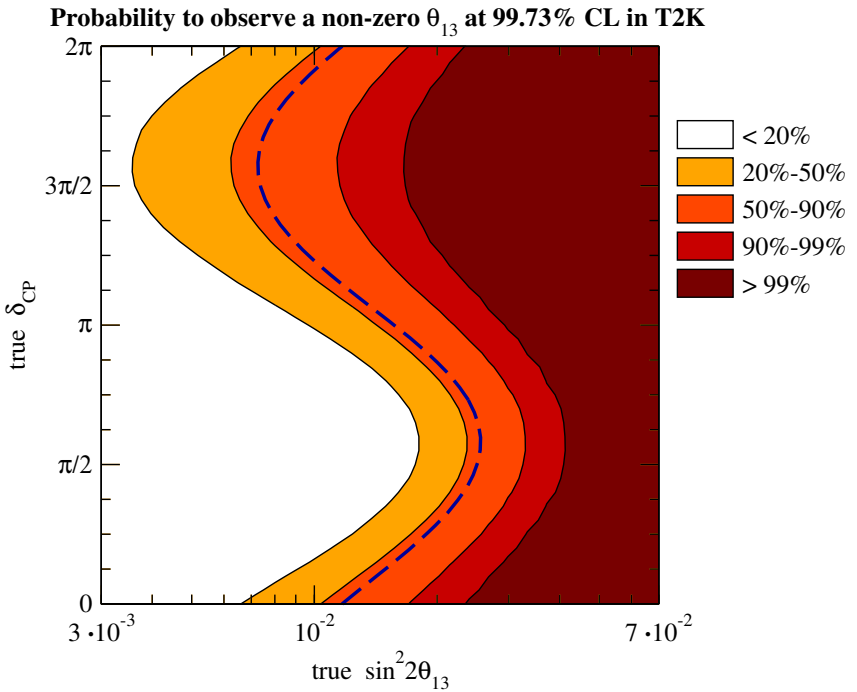
- Define the test statistic "$\Delta\chi^2$"

$$\Delta\chi^2(\theta) = -2\log\left[\frac{\mathcal{L}(\theta|d)}{\sup_{\theta'}\mathcal{L}(\theta'|d)}\right]$$

- Assume it is χ^2 distributed with n degrees of freedom
- Use the data set without statistical fluctuations (Asimov data)
- Quote result

The interpretation

- $\Delta\chi^2$ *is* asymptotically χ^2 (Wilks' theorem)
- The Asimov data is representative
- Works for *nested hypotheses*
- Some additional requirements



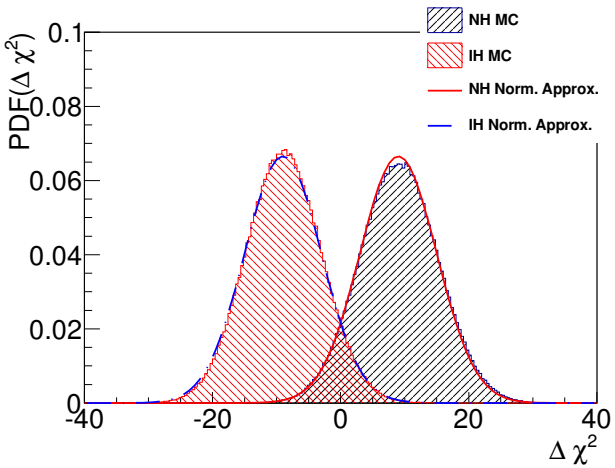
Schwetz, Phys.Lett. B648 (2007) 54

Not a nested hypothesis

- Mass ordering is not nested
- Wilks' theorem not applicable
- Some different choices of test statistic

$$\begin{aligned} \Delta\chi^2 &= \chi^2_{\mathrm{NO}} - \min \chi^2 \\ T &= \chi^2_{\mathrm{IO}} - \chi^2_{\mathrm{NO}} \\ T' &= \chi^2(\theta) - \min \chi^2 \end{aligned}$$

  - All have different distributions
  - We concentrate on T



Qian, *et al.*, Phys.Rev. D86 (2012) 113011

$$T \simeq \mathcal{N}(T_0, 2\sqrt{T_0})$$

$T_0 =$ value for Asimov data

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Back to basics

- Hypothesis testing
- Test hypothesis H_0, alternative hypothesis H_1
- Critical region defined by T_c
- Reject H_0 if $T < T_c$
- Test significance: $1 - \alpha = 1 - P(T < T_c|H_0)$
- Test power: $1 - \beta = P(T < T_c|H_1)$
- *Both* α and β are relevant

What is sensitivity?

Sensitivity (median)

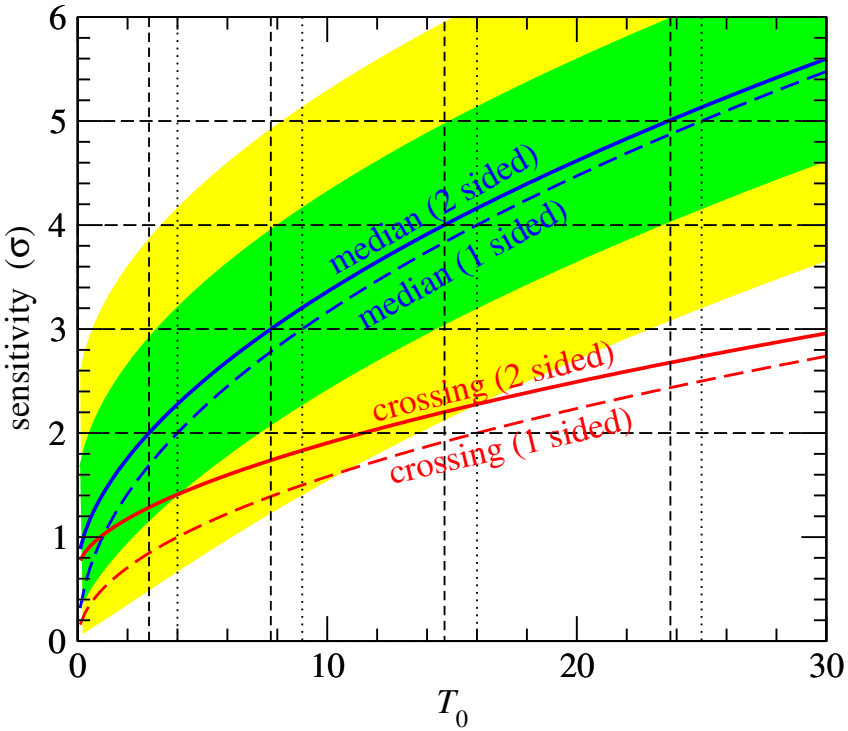
What is the expected rejection of a false hypothesis? (Given a particular true hypothesis)

Other

- What is the probability of ruling out a false hypothesis at $x\sigma$?
- At what CL is exactly one hypothesis ruled out?

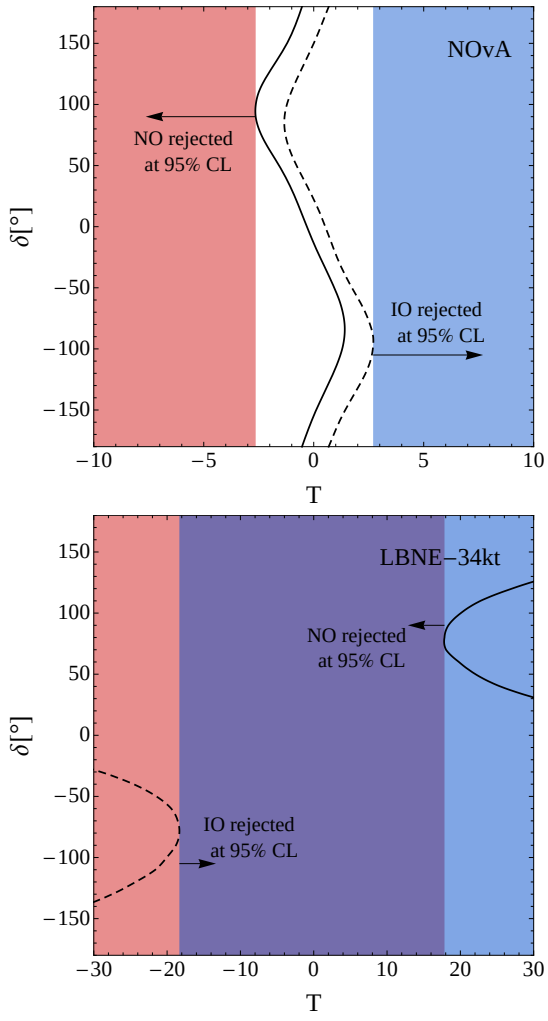
Simple hypotheses

- Hypotheses are not parameter dependent
- Nyman-Pearson lemma applies
- Straight forward
- Applicable to reactor experiments



MB, Coloma, Huber, Schwetz, JHEP 03(2014)028

Composite hypotheses



MB, Coloma, Huber, Schwetz, JHEP 03(2014)028

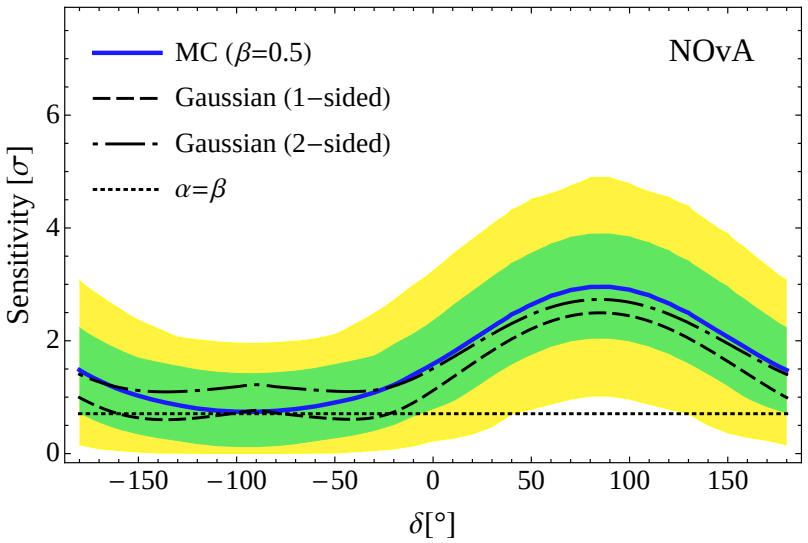
- Hypotheses that have parameter dependence
- To reject = Rejecting all parameter sets
- Distribution of T is parameter dependent

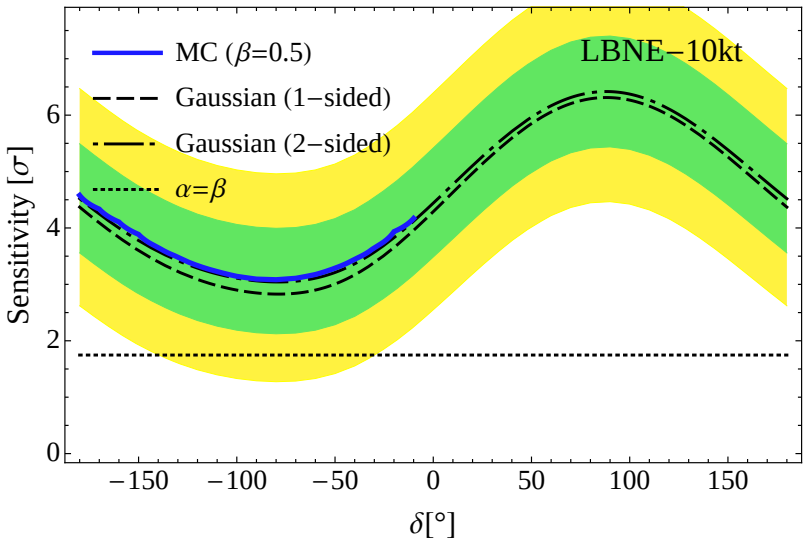
$$T_c = \min_\theta T_c(\theta)$$

- Extensive Monte Carlo simulations
- Approximation (must check validity)

$$T(\theta) = \mathcal{N}(T_0(\theta), 2\sqrt{T_0(\theta)})$$

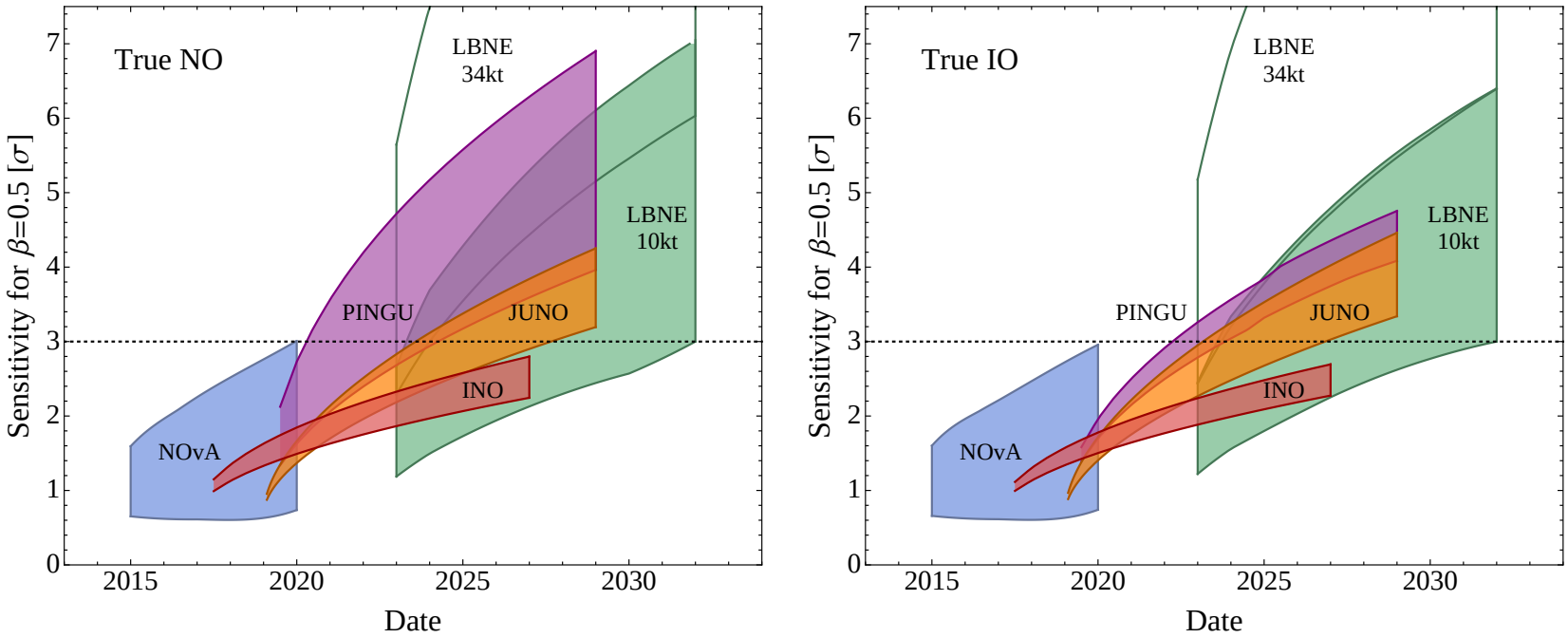
Accelerator experiments - results





MB, Coloma, Huber, Schwetz, JHEP 03(2014)028

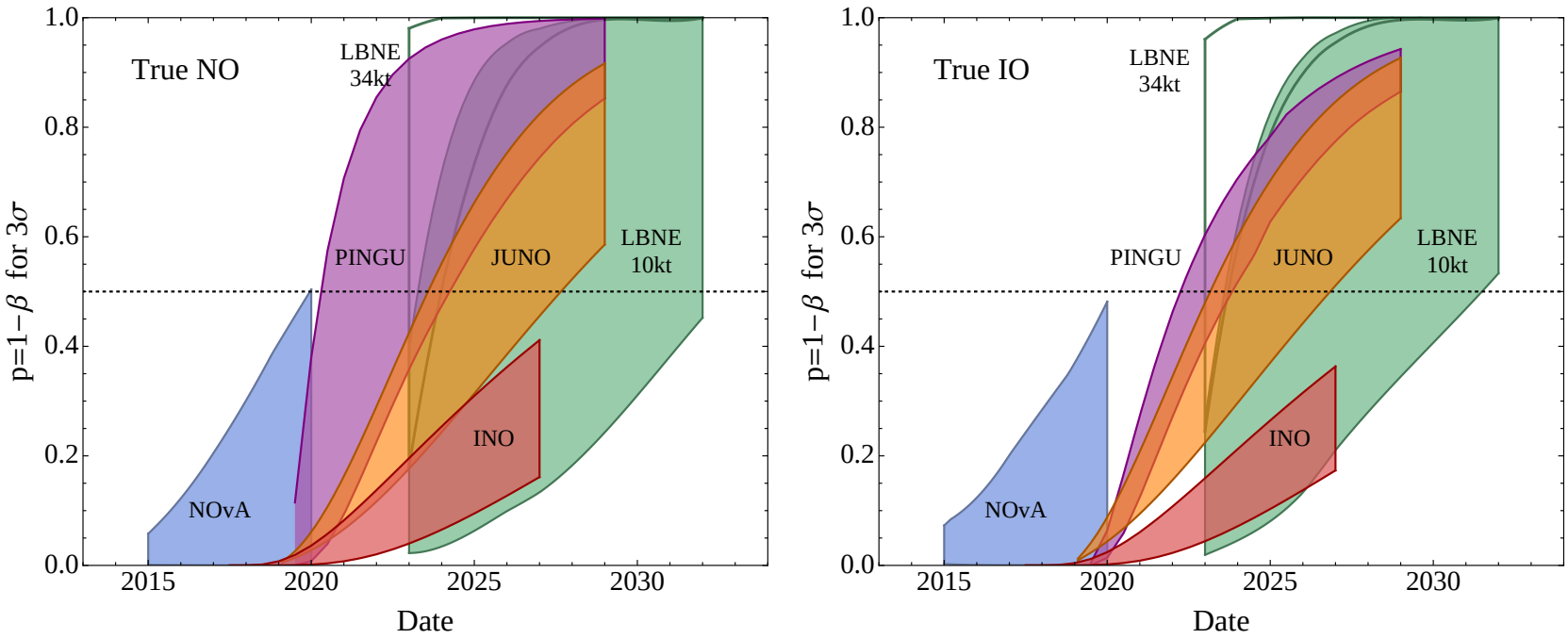
Comparison of experiments - specific β



MB, Coloma, Huber, Schwetz, JHEP 03(2014)028

Note: Bands have different meanings!

Comparison of experiments - specific α



MB, Coloma, Huber, Schwetz, JHEP 03(2014)028

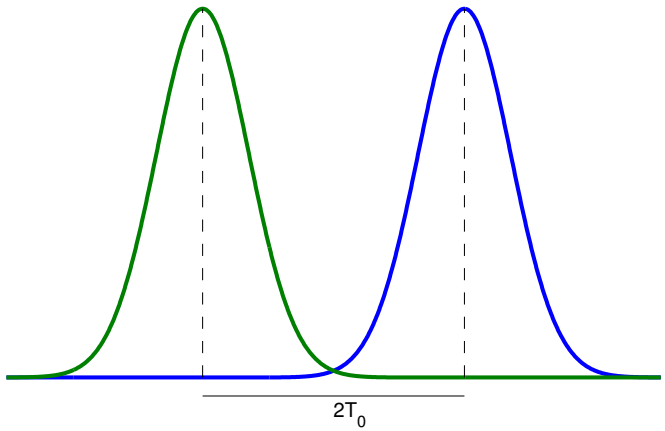
Note: Bands have different meanings!

At the end of the day

For the simple hypotheses:

- Two Gaussians, $H_\pm : \mathcal{N}(\pm T_0, 2\sqrt{T_0})$
- For H_+, *typical* (median) result is $+T_0$
- $+T_0$ is $T_0 - (-T_0) = 2T_0$ away from the expected H_- result
- $2T_0/(2\sqrt{T_0}) = \sqrt{T_0}$

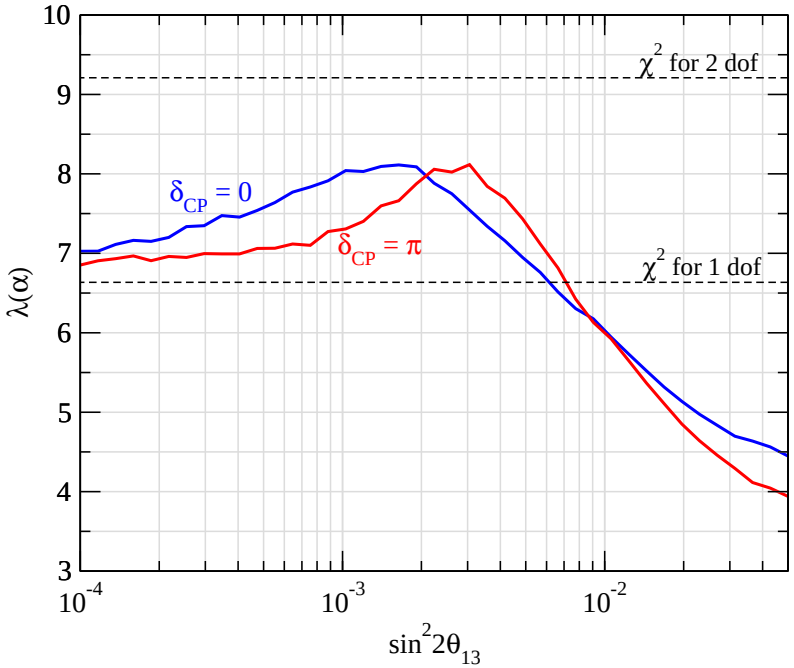
See also: Vitelis, Read, 1311.4076



How to interpret the median sensitivity

- It is *representative* for how well the experiment will do
- 50 % probability of not reaching it
- 50 % probability of *doing better*
- *Not* 50 % probability of "being wrong"
- Not the only relevant quantity, distribution matters (do Brazilian bands!)
- Personal preference: Quote the power $1 - \beta$ for a target sensitivity

What about δ_{CP}?



Schwetz, Phys.Lett. B648 (2007) 54

- The test statistic may be very different from a χ^2 distribution
- Checked with full Feldman–Cousins approach

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Bayesian hypothesis testing

- Study the relative degrees of belief in two hypotheses

$$\frac{P(H_1;d)}{P(H_2;d)} = \frac{\mathcal{L}_{H_1}(d)}{\mathcal{L}_{H_2}(d)} \frac{P(H_1)}{P(H_2)}$$

- Strength of the evidence for H_1:

$$\kappa = 2 \log \left[\frac{P(H_1;d)}{P(H_2;d)} \right]$$

- Kass-Raftery scale:

Strength of evidence for H	κ	Posterior odds	Degree of belief
Barely worth mentioning	0 to 2	ca 1 to 3	$< 73.11\%$
Positive	2 to 6	ca 3 to 20	$> 73.11\%$
Strong	6 to 10	ca 20 to 150	$> 95.26\%$
Very strong	> 10	$\gtrsim 150$	$> 99.33\%$

What can be said about the future?

- Can compute probability of obtaining evidence at least strength κ_0 *for the true ordering*

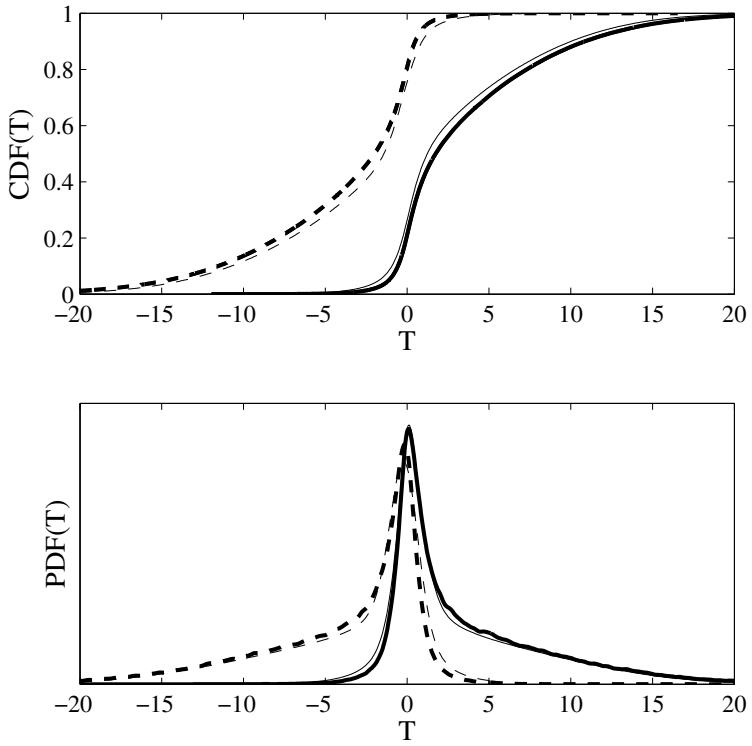
$$P(\kappa_0) = P(\kappa > \kappa_0; H_1)P(H_1) + P(\kappa < -\kappa_0; H_2)P(H_2)$$

- Typical choice $P(H_1) = P(H_2) = 0.5$
- Takes into account information on oscillation parameters

$$\mathcal{L}_H(d) = \int \mathcal{L}_{H(\theta)}(d)\pi(\theta)d\theta$$

- Compactifies all of the available information to one number
- Easy to simulate through Monte Carlo methods
- Prior dependent

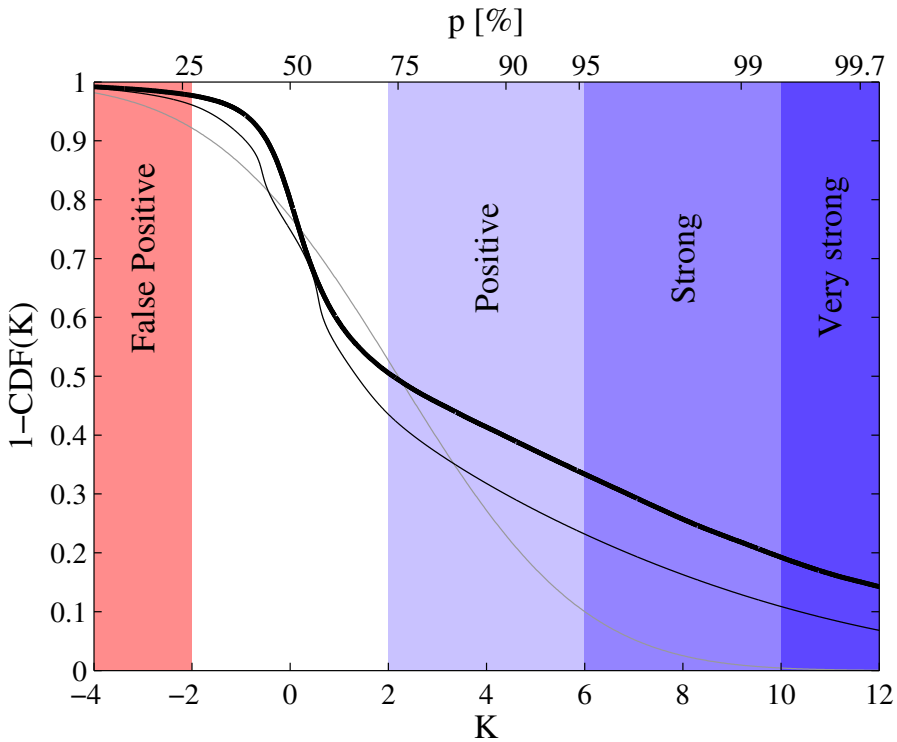
Example: NOνA



MB, JHEP 01(2014)139

- NOνA experiment
- GLoBES implementation
- Only δ and Δm^2_{31} varying
- Flat and 10 % Gaussian priors, respectively

Example: NOνA, results



MB, JHEP 01(2014)139

- Probability of obtaining evidence with *at least* strength K *for the correct ordering*

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Summary and conclusions

- Wilks' theorem is not applicable, the test statistic is not χ^2 distributed
- Regardless, $\sqrt{T_0}$ is still a good approximation of the (median) sensitivity
- Frequentist methods are perfectly applicable
- Bayesian methods equally applicable, matter of preference
- Important not to mix the concepts and be aware of the proper interpretation

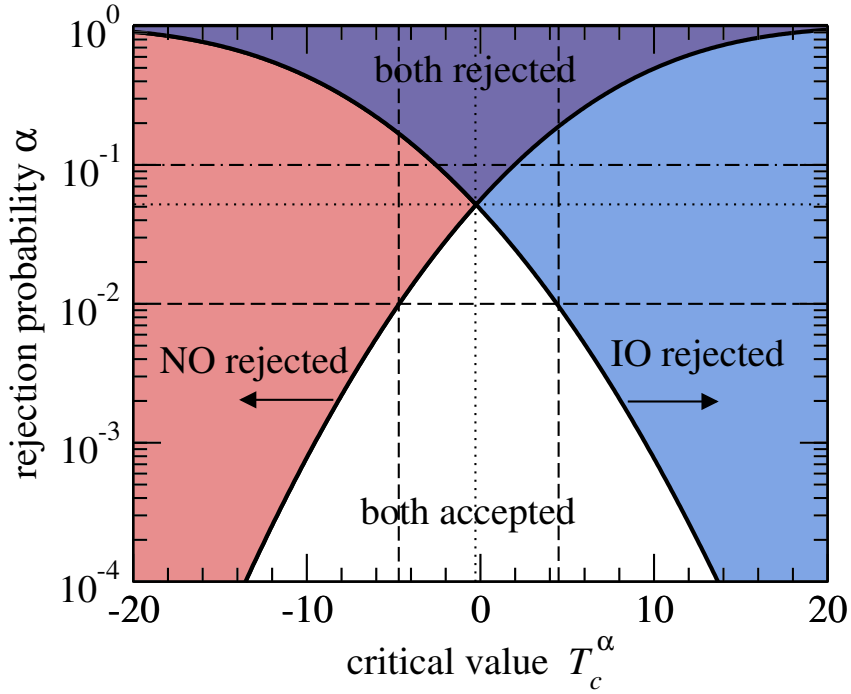
5 Backup frequentist

6 Backup distributions and results

7 Backup Bayesian basics

Testing both orderings

- Depending on the significance it may be possible to:
  - Reject exactly one hypothesis
  - Reject both hypotheses
  - Not reject any hypothesis
- It is natural, the true hypotheses *should* be rejected with probability α



MB, Coloma, Huber, Schwetz, JHEP 03(2014)028

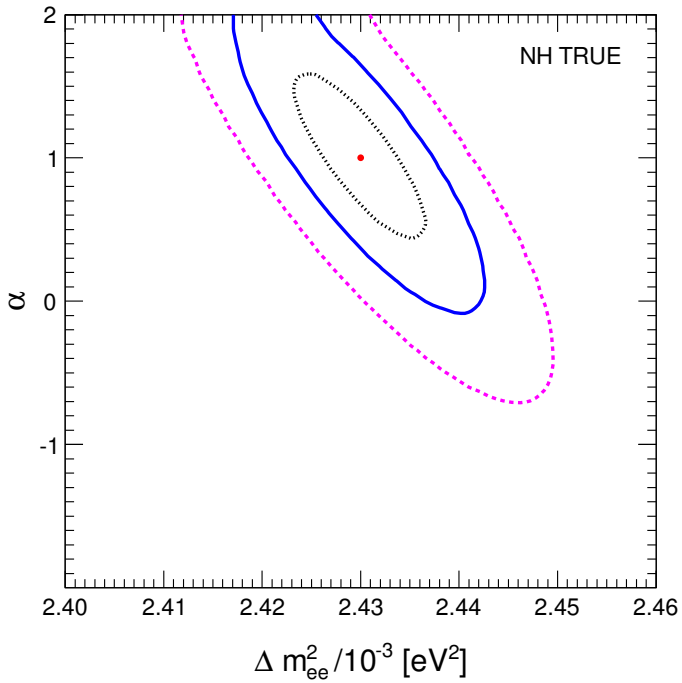
Interpolating parameters

- Introduce an interpolating continuous parameter α such that

$$P(\alpha = \pm 1, \theta) = P(\theta, \mathrm{NO/IO})$$

see, e.g., Capozzi, Lisi, Marrone, arXiv:1309.1638

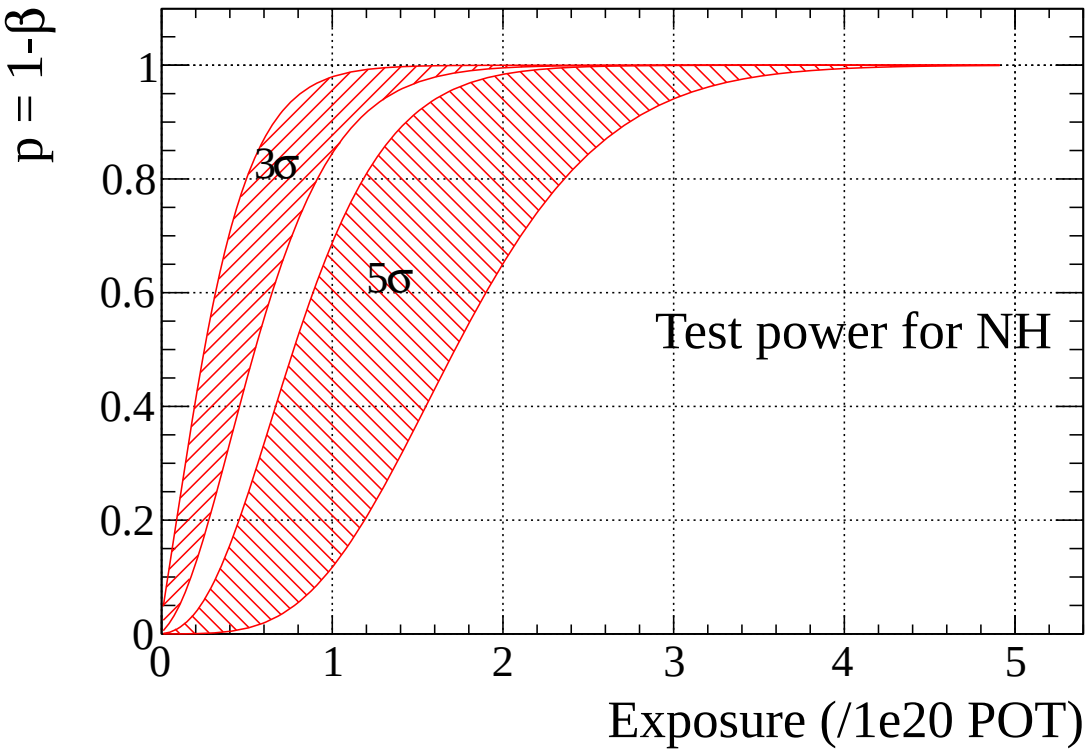
- Wilks' theorem now applies
- Put bounds on α
- If $\alpha = 1$ (-1) is allowed, NO (IO) is allowed



Capozzi, Lisi, Marrone, arXiv:1309.1638

- Personal comment: $\alpha = 0$ is not special, it being allowed a priori does not affect the rejection of NO/IO

LBNO predictions



LBNO collaboration, arXiv:1312.6520

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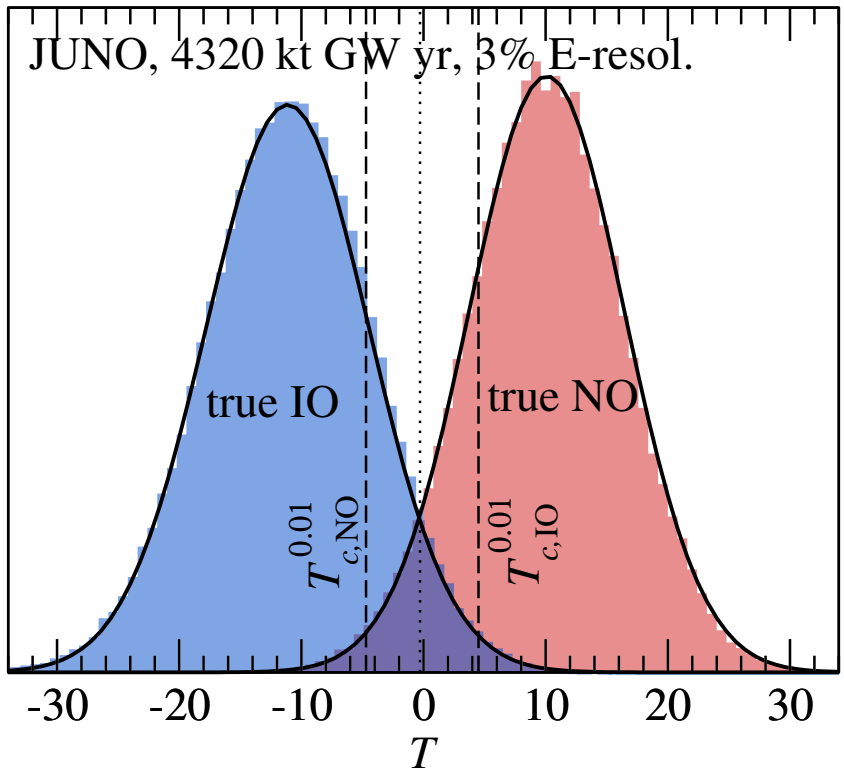
Reactor experiments

- Distribution of T essentially parameter independent
- Simple hypotheses
- Gaussian limit well satisfied
- Median sensitivity $n\sigma$:

$$n \simeq \sqrt{2}\,\mathrm{erfc}^{-1}\left[\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{T_0}{2}}\right)\right]$$



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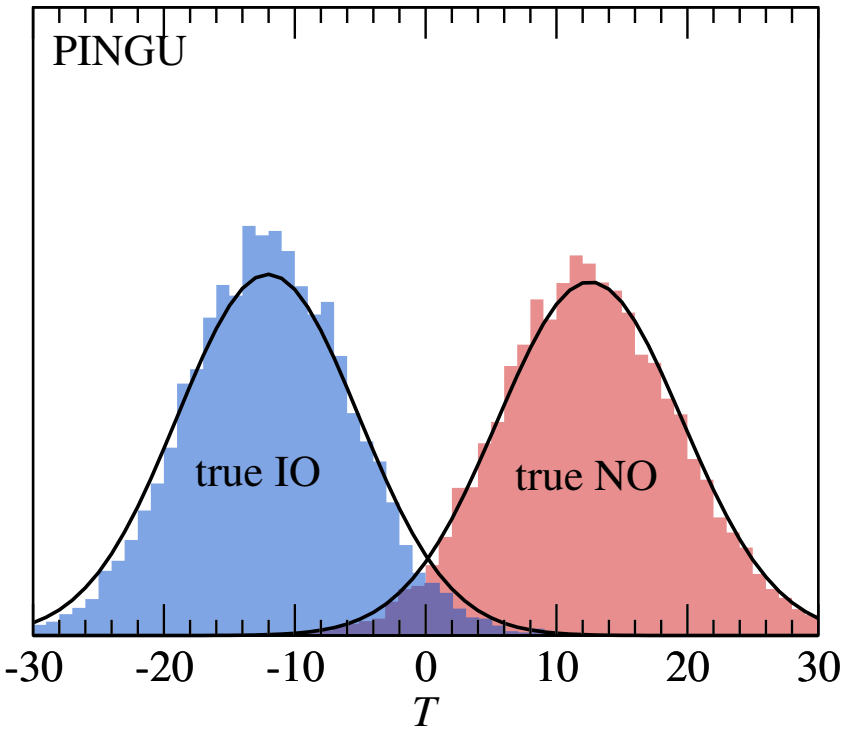
Atmospheric experiments



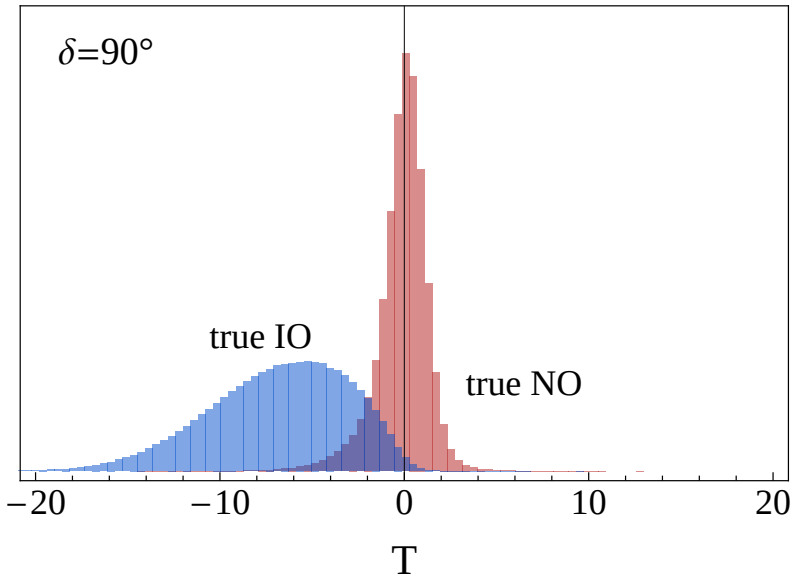
MB, Coloma, Huber, Schwetz, JHEP 03(2014)028

- Distribution of T mainly depends on θ_{23}
- Gaussianity is still well satisfied for each θ_{23}
- Analytic as function of $T_0(\theta_{23})$
- Boils down to evaluating the Asimov data: T_0

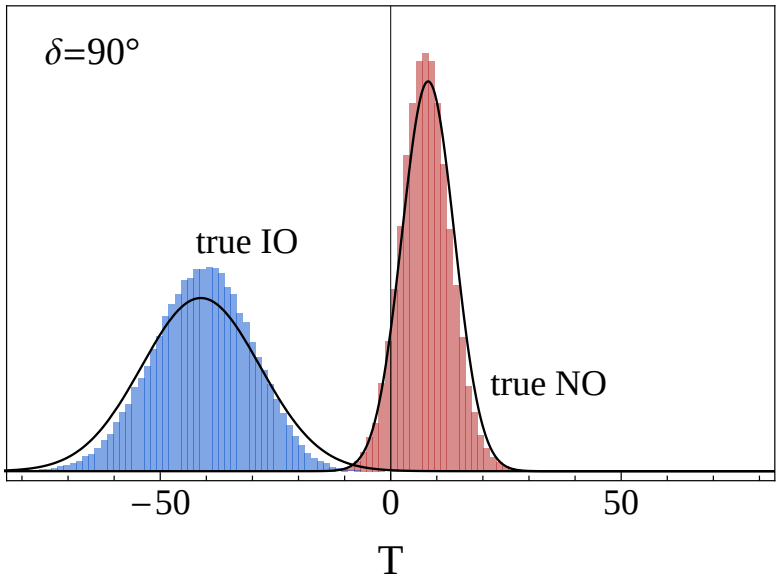
Accelerator experiments - distributions

- Not always Gaussian!
- Typical for low-sensitivity experiments
- Need to perform Monte Carlo studies for accuracy
- Rejection power *depends on the true parameters*

NOνA



LBNE



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Bayesian basics

- Assign a degree of belief in each hypothesis $P(H)$
- Update the degree of belief depending on observations
- Bayes' theorem

$$P(A,B) = P(A;B)P(B) = P(B;A)P(A)$$

$$P(A;B) = \frac{P(B;A)P(A)}{P(B)}$$

- Take $A =$ hypothesis H, $B =$ data d

$$P(H;d) = \frac{\mathcal{L}_H(d)P(H)}{P(d)}$$